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Moments of Generalized Quadratic Gauss Sums Weighted by L-Functions[1]

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The main purpose of this paper is using estimates for character sums and analytic methods to study the second, fourth, and sixth order moments of generalized quadratic Gauss sums weighted by L-functions. Three asymptotic formulae are obtained.

Key Words: general quadratic Gauss sums; L-functions; asymptotic formula.

1. INTRODUCTION

Let $q \geqslant 2$ be an integer; χ denotes a Dirichlet character modulo q. For any integer n, we define the general quadratic Gauss sums $G(n, \chi; q)$ as

$$G(n, \chi; q) = \sum_{a=1}^{q} \chi(a)\, e\left(\frac{na^2}{q}\right),$$

where $e(y) = e^{2\pi i y}$. This sum is important because it is a generalization of the classical quadratic Gauss sum. But about the properties of $G(n, \chi; q)$, we know very little at present. The value of $|G(n, \chi; q)|$ is irregular as χ varies. One can only get some upper bound estimates. For example, for any integer n with $(n, q) = 1$, from the general result of Cochrane and Zheng [8] we can deduce

$$|G(n, \chi; q)| \leqslant 2^{\omega(q)} q^{\frac{1}{2}},$$

where $\omega(q)$ denotes the number of distinct prime divisors of q; the case where q is prime is due to Weil [9].

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In this paper we show that $G(n, \chi; p)$ enjoys many good weighted mean value properties. For convenience, in the following we always suppose that p denotes an odd prime, $L(s, \chi)$ the Dirichlet L-function corresponding to the character $\chi \bmod p$, and that $G(n; q)$ denotes the classical quadratic Gauss sum. We use estimates for character sums and the analytic methods to prove the following three results:

THEOREM 1. *For any integer n with $(n, p)=1$, we have the asymptotic formula*

$$\sum_{\chi \neq \chi_0} |G(n, \chi; p)|^2 \cdot |L(1, \chi)| = C \cdot p^2 + O(p^{\frac{3}{2}} \ln^2 p),$$

where

$$C = \prod_p \left[1 + \frac{\binom{2}{1}^2}{4^2 \cdot p^2} + \frac{\binom{4}{2}^2}{4^4 \cdot p^4} + \cdots + \frac{\binom{2m}{m}^2}{4^{2m} \cdot p^{2m}} + \cdots \right]$$

is a constant, $\sum_{\chi \neq \chi_0}$ denotes the summation over all nonprincipal characters modulo p, $\prod_p$ denotes the product over all primes, and $\binom{2m}{m} = (2m)!/(m!)^2$.

THEOREM 2. *For any integer n with $(n, p) = 1$, we have the asymptotic formula*

$$\sum_{\chi \neq \chi_0} |G(n, \chi; p)|^4 \cdot |L(1, \chi)| = 3 \cdot C \cdot p^3 + O(p^{\frac{5}{2}} \ln^2 p).$$

THEOREM 3. *Let p be an odd prime with $p \equiv 3 \bmod 4$. Then for any fixed positive integer n with $(n, p) = 1$, we have the asymptotic formula*

$$\sum_{\chi \neq \chi_0} |G(n, \chi; p)|^6 \cdot |L(1, \chi)| = 10 \cdot C \cdot p^4 + O(p^{\frac{7}{2}} \ln^2 p).$$

Let n be any integer with $(n, p) = 1$. Then using our methods we can easily deduce the identities

$$\sum_{\chi \bmod p} |G(n, \chi; p)|^4 = \begin{cases} (p-1)\left[3p^2 - 6p - 1 + 4\left(\frac{n}{p}\right)\sqrt{p}\right], & \text{if} \quad p \equiv 1 \bmod 4; \\ (p-1)(3p^2 - 6p - 1), & \text{if} \quad p \equiv 3 \bmod 4, \end{cases}$$

and

$$\sum_{\chi \bmod p} |G(n, \chi; p)|^6 = (p-1)(10p^3 - 25p^2 - 4p - 1), \qquad \text{if} \quad p \equiv 3 \bmod 4,$$

where $(\frac{n}{p})$ is the Legendre symbol.

For a general integer $m \geqslant 3$, whether there exists an asymptotic formula for

$$\sum_{\chi \bmod p} |G(n, \chi; p)|^{2m} \qquad \text{and} \qquad \sum_{\chi \neq \chi_0} |G(n, \chi; p)|^{2m} \, |L(1, \chi)|$$

is an unsolved problem. We believe that it is true and that we even have the following.

Conjecture. For all positive integer m,

$$\sum_{\chi \neq \chi_0} |G(n, \chi; p)|^{2m} \, |L(1, \chi)| \sim C \cdot \sum_{\chi \bmod p} |G(n, \chi; p)|^{2m}, \qquad p \to +\infty,$$

where C is the same as in Theorem 1.

2. SOME LEMMAS

In order to complete the proof of the theorems, we need the following lemmas.

LEMMA 1. *For any odd prime p, we have the estimate*

$$\sum_{a=1}^{p-1} \left| \sum_{\chi \neq \chi_0} \chi(a) \, |L(1, \chi)| \right| = O(p \ln p).$$

Proof. Let $N = p^{3/2}$, χ be a nonprincipal character mod p, and $A(\chi, y) = \sum_{N < n \leqslant y} \chi(n)$. Then by Abel's identity and the Pólya–Vinogradov inequality we have

$$\begin{aligned} L(1, \chi) &= \sum_{n \leqslant N} \frac{\chi(n)}{n} + \int_N^{+\infty} \frac{A(\chi, y)}{y^2} \, dy \\ &= \sum_{n \leqslant N} \frac{\chi(n)}{n} + O\left(\frac{\ln p}{p}\right). \end{aligned}$$

So that

$$(1) \qquad |L(1, \chi)| = \left| \sum_{n \leqslant N} \frac{\chi(n)}{n} \right| + O\left(\frac{\ln p}{p}\right).$$

On the other hand, let $r(n)$ be a multiplicative function defined by

$$r(p^\alpha) = \frac{\binom{2\alpha}{\alpha}}{4^\alpha} \qquad \text{and} \qquad r(1) = 1,$$

where p is a prime and α is any positive integer. For this number-theoretic function $r(n)$, it is easily proved that (see Lemma 1 of [6])

$$\sum_{d \mid n} r(d) \cdot r\left(\frac{n}{d}\right) = 1$$

and

$$(2) \qquad \left(\sum_{n \leqslant N} \frac{\chi(n)\, r(n)}{n} \right)^2 = \sum_{m \leqslant N} \sum_{n \leqslant N} \frac{\chi(nm)\, r(m)\, r(n)}{mn}$$
$$= \sum_{n \leqslant N} \frac{\chi(n)}{n} + \sum_{N < n \leqslant N^2} \frac{\chi(n)\, r(n, N)}{n},$$

where

$$r(n, N) = \sum_{\substack{d \mid n \\ d, \frac{n}{d} \leqslant N}} r(d) \cdot r\left(\frac{n}{d}\right).$$

From (2), Cauchy's inequality, and the orthogonality relationships for character sums

$$\sum_{\chi \bmod p} \chi(n) = \begin{cases} p-1, & \text{if } \quad n \equiv 1 \bmod p; \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\sum_{a=1}^{p-1} \chi(a) = \begin{cases} p-1, & \text{if } \quad \chi = \chi_0; \\ 0, & \text{otherwise,} \end{cases}$$

we have the estimates

$$
\begin{aligned}
(3)\qquad &\sum_{a=1}^{p-1}\left|\sum_{\chi}\chi(a)\left(\left|\sum_{n\leqslant N}\frac{\chi(n)}{n}\right|-\left|\sum_{n\leqslant N}\frac{\chi(n)\,r(n)}{n}\right|^{2}\right)\right| \\
&\leqslant p^{1/2}\cdot\left[\sum_{a=1}^{p-1}\left|\sum_{\chi}\chi(a)\left(\left|\sum_{n\leqslant N}\frac{\chi(n)}{n}\right|-\left|\sum_{n\leqslant N}\frac{\chi(n)\,r(n)}{n}\right|^{2}\right)\right|^{2}\right]^{\frac{1}{2}} \\
&\leqslant p\cdot\left[\sum_{\chi}\left(\left|\sum_{n\leqslant N}\frac{\chi(n)}{n}\right|-\left|\sum_{n\leqslant N}\frac{\chi(n)}{n}+\sum_{N<n\leqslant N^2}\frac{\chi(n)\,r(n,N)}{n}\right|\right)^{2}\right]^{\frac{1}{2}} \\
&\leqslant p\cdot\left[\sum_{\chi}\left|\sum_{N<n\leqslant N^2}\frac{\chi(n)\,r(n,N)}{n}\right|^{2}\right]^{\frac{1}{2}} \\
&\leqslant p^{3/2}\cdot\left(\sum_{N<m\leqslant N^2}\ \sum_{\substack{N<n\leqslant N^2\\ m\equiv n \bmod p}}\frac{r(m,N)\cdot r(n,N)}{mn}\right)^{\frac{1}{2}} \\
&\ll p\cdot\ln p
\end{aligned}
$$

and

$$
\begin{aligned}
(4)\qquad \sum_{a=1}^{p-1}\left|\sum_{\chi}\chi(a)\left|\sum_{n\leqslant N}\frac{\chi(n)\,r(n)}{n}\right|^{2}\right| &= (p-1)\sum_{a=1}^{p-1}\ \sum_{m\leqslant N}\ \sum_{\substack{n\leqslant N\\ am\equiv n \bmod p}}\frac{r(m)\,r(n)}{mn} \\
&= (p-1)\cdot\left(\sum_{m\leqslant N}\frac{r(m)}{m}\right)^{2}\ll p\ln p.
\end{aligned}
$$

Combining (1), (3), and (4) we obtain

$$
\begin{aligned}
&\sum_{a=1}^{p-1}\left|\sum_{\chi\neq\chi_0}\chi(a)\,|L(1,\chi)|\right| \\
&\qquad=\sum_{a=1}^{p-1}\left|\sum_{\chi\neq\chi_0}\chi(a)\left|\sum_{n\leqslant N}\frac{\chi(n)}{n}\right|+O(\ln p)\right| \\
&\qquad=\sum_{a=1}^{p-1}\left|\sum_{\chi}\chi(a)\left|\sum_{n\leqslant N}\frac{\chi(n)}{n}\right|\right|+O(p\ln p) \\
&\qquad\ll\sum_{a=1}^{p-1}\left|\sum_{\chi}\chi(a)\left(\left|\sum_{n\leqslant N}\frac{\chi(n)}{n}\right|-\left|\sum_{n\leqslant N}\frac{\chi(n)\,r(n)}{n}\right|^{2}\right)\right| \\
&\qquad\quad+\sum_{a=1}^{p-1}\left|\sum_{\chi}\chi(a)\left|\sum_{n\leqslant N}\frac{\chi(n)\,r(n)}{n}\right|^{2}\right|+p\ln p \\
&\qquad\ll p\cdot\ln p.
\end{aligned}
$$

This proves Lemma 1. ▮

LEMMA 2. *For any odd prime p, we have the asymptotic formula*

$$\sum_{\chi(-1)=1}' |L(1,\chi)| = \tfrac{1}{2}\cdot C\cdot p + O(p^{\frac{1}{2}}\cdot \ln p),$$

where

$$C = \sum_{n=1}^{\infty} \frac{r^2(n)}{n^2} = \prod_p \left[1 + \frac{\binom{2}{1}^2}{4^2\cdot p^2} + \frac{\binom{4}{2}^2}{4^4\cdot p^4} + \cdots + \frac{\binom{2n}{n}^2}{4^{2n}\cdot p^{2n}} + \cdots \right]$$

is an absolute constant, and $\sum'_{\chi(-1)=1}$ *denotes the summation over all nonprincipal even characters mod* p.

Proof. Let $N = p^{3/2}$. Note the orthogonality relationship for character sums

$$\sum_{\chi(-1)=1} \chi(n) = \begin{cases} \frac{1}{2}(p-1), & \text{if } \; n \equiv \pm 1 \bmod p; \\ 0, & \text{otherwise.} \end{cases}$$

By (1), (2), and the method of proving Lemma 1 we have

$$\begin{aligned}
\sum_{\chi(-1)=1}' |L(1,\chi)| &= \sum_{\chi(-1)=1}' \left| \sum_{n\leqslant N} \frac{\chi(n)}{n} \right| + O(\ln p) \\
&= \sum_{\chi(-1)=1} \left| \sum_{n\leqslant N} \frac{\chi(n)\, r(n)}{n} \right|^2 + O(\ln p) \\
&\quad + \sum_{\chi(-1)=1} \left(\left| \sum_{n\leqslant N} \frac{\chi(n)}{n} \right| - \left| \sum_{n\leqslant N} \frac{\chi(n)\, r(n)}{n} \right|^2 \right) \\
&= \frac{1}{2}(p-1)\cdot \sum_{\substack{m\leqslant N \\ m\equiv \pm n}} \sum_{\substack{n\leqslant N \\ \bmod p}} \frac{r(m)\cdot r(n)}{mn} + O(\ln p) \\
&\quad + O\left(\sum_{\chi} \left| \sum_{N<n\leqslant N^2} \frac{\chi(n)\cdot r(n,N)}{n} \right| \right) \\
&= \frac{1}{2} p\cdot \sum_{\substack{n=1 \\ (n,p)=1}}^{\infty} \frac{r^2(n)}{n^2} + O(\ln^2 p) \\
&\quad + O\left(p^{1/2} \left(\sum_{\chi} \left| \sum_{N<n\leqslant N^2} \frac{\chi(n)\cdot r(n,N)}{n} \right|^2 \right)^{\frac{1}{2}} \right)
\end{aligned}$$

$$= \frac{1}{2} p \cdot \sum_{n=1}^{\infty} \frac{r^2(n)}{n^2} + O(p^{\frac{1}{2}} \cdot \ln p)$$

$$= \frac{1}{2} \cdot C \cdot p + O(p^{\frac{1}{2}} \cdot \ln p).$$

This proves Lemma 2. ∎

LEMMA 3. *For any integer $q \geqslant 1$, we have the formula*

$$G(1; q) = \tfrac{1}{2} \sqrt{q}\,(1+i)(1+e^{\frac{-\pi i q}{2}}) = \begin{cases} \sqrt{q} & \textit{if} \quad q \equiv 1 \pmod 4; \\ 0 & \textit{if} \quad q \equiv 2 \pmod 4; \\ i\sqrt{q} & \textit{if} \quad q \equiv 3 \pmod 4; \\ (1+i)\sqrt{q} & \textit{if} \quad q \equiv 0 \pmod 4). \end{cases}$$

Proof. This is a remarkable formula of Gauss. See Theorem 9.16 of [1]. ∎

LEMMA 4. *Let p be an odd prime, χ be any nonprincipal even character (i.e., $\chi(-1) = 1$ and $\chi \neq \chi_0$) mod p. Then for any integer n with $(n, p) = 1$, we have the identity*

$$|G(n, \chi; p)|^2 = 2p + \left(\frac{n}{p}\right) G(1; p) \sum_{a=1}^{p-1} \chi(a) \left(\frac{a^2-1}{p}\right),$$

where $(\frac{n}{p})$ is the Legendre symbol.

Proof. First we note that if $p \nmid n$, then (See Theorem 7.5.4 of [7])

$$G(n; p) = \left(\frac{n}{p}\right) G(1; p). \tag{5}$$

From (5) we know that if χ is a nonprincipal even character mod p, then

$$|G(n, \chi; p)|^2 = \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi(a)\,\bar{\chi}(b)\, e\left(\frac{na^2 - nb^2}{p}\right)$$

$$= \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi(a\bar{b})\, e\left(\frac{n(a^2-b^2)}{p}\right)$$

$$= \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi(a)\, e\left(\frac{nb^2(a^2-1)}{p}\right)$$

$$= \sum_{a=1}^{p-1} \chi(a) \left[\sum_{b=1}^{p} e\left(\frac{nb^2(a^2-1)}{p} \right) - 1 \right]$$

$$= 2p + \sum_{a=2}^{p-2} \chi(a)\, G(n(a^2-1); p) - \sum_{a=1}^{p-1} \chi(a)$$

$$= 2p + \left(\frac{n}{p} \right) G(1; p) \sum_{a=1}^{p-1} \chi(a) \left(\frac{a^2-1}{p} \right).$$

This proves Lemma 4.

3. PROOF OF THE THEOREMS

In this section, we shall complete the proof of the theorems. We only prove Theorem 2 and Theorem 3, the proof of Theorem 1 being similar. Note that if χ is an odd character modulo p, then

$$G(n, \chi; p) = \sum_{a=1}^{p} \chi(a)\, e\left(\frac{a^2 n}{p} \right) = 0.$$

Thus for any integer n with $(n, p) = 1$, from Lemma 4 we have

$$\sum_{\chi \neq \chi_0} |G(n, \chi; p)|^4 \cdot |L(1, \chi)| \tag{6}$$

$$= \sum_{\substack{\chi \neq \chi_0 \\ \chi(-1)=1}} |G(n, \chi; p)|^4 \cdot |L(1, \chi)|$$

$$= \sum_{\substack{\chi \neq \chi_0 \\ \chi(-1)=1}} \left[2p + \left(\frac{n}{p} \right) G(1; p) \sum_{a=1}^{p-1} \chi(a) \left(\frac{a^2-1}{p} \right) \right]^2 \cdot |L(1, \chi)|$$

$$= \sum_{\substack{\chi \neq \chi_0 \\ \chi(-1)=1}} \left[4p^2 + 4p \left(\frac{n}{p} \right) G(1; p) \sum_{a=1}^{p-1} \chi(a) \left(\frac{a^2-1}{p} \right) \right] \cdot |L(1, \chi)|$$

$$+ G^2(1; p) \cdot \sum_{\substack{\chi \neq \chi_0 \\ \chi(-1)=1}} \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi(ab) \left(\frac{a^2-1}{p} \right) \left(\frac{b^2-1}{p} \right) \cdot |L(1, \chi)|.$$

Note the identities

$$(7)\qquad \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(ab)\left(\frac{a^2-1}{p}\right)\left(\frac{b^2-1}{p}\right)$$
$$=\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a)\left(\frac{a^2\bar{b}^2-1}{p}\right)\left(\frac{b^2-1}{p}\right)$$
$$=\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a)\left(\frac{a^2-b^2}{p}\right)\left(\frac{b^2-1}{p}\right)$$
$$=2\left(\frac{-1}{p}\right)(p-3)+\sum_{a=2}^{p-2}\sum_{b=1}^{p-1}\chi(a)\left(\frac{a^2-b^2}{p}\right)\left(\frac{b^2-1}{p}\right)$$

and

$$\sum_{a=2}^{p-2}\left(\frac{a^2-b^2}{p}\right)\sum_{\chi(-1)=-1}\chi(a)\,|L(1,\chi)|$$
$$=\sum_{a=1}^{p-1}\left(\frac{a^2-1}{p}\right)\sum_{\chi(-1)=-1}\chi(a)\,|L(1,\chi)|=0,$$

and the estimate of Weil

$$(8)\qquad \sum_{b=1}^{p-1}\left(\frac{b^2-a^2}{p}\right)\left(\frac{b^2-1}{p}\right)\leqslant 3\sqrt{p},\qquad a^2\not\equiv 1 \mod p.$$

From (6), (7), (8), Lemma 1, Lemma 2, and Lemma 3 we have

$$\sum_{\chi\neq\chi_0}|G(n,\chi;p)|^4\cdot|L(1,\chi)|$$
$$=\left(4p^2+G^2(1;p)\cdot 2\left(\frac{-1}{p}\right)(p-3)\right)\cdot\sum_{\substack{\chi\neq\chi_0\\ \chi(-1)=1}}|L(1,\chi)|$$
$$+4p\left(\frac{n}{p}\right)G(1;p)\sum_{a=1}^{p-1}\left(\frac{a^2-1}{p}\right)\sum_{\substack{\chi\neq\chi_0\\ \chi(-1)=1}}\chi(a)\,|L(1,\chi)|$$
$$+G^2(1;p)\sum_{a=2}^{p-2}\sum_{b=1}^{p-1}\left(\frac{a^2-b^2}{p}\right)\left(\frac{b^2-1}{p}\right)\sum_{\substack{\chi\neq\chi_0\\ \chi(-1)=1}}\chi(a)\,|L(1,\chi)|$$
$$=(6p^2-6p)\sum_{\substack{\chi\neq\chi_0\\ \chi(-1)=1}}|L(1,\chi)|+O\left(p^{3/2}\sum_{a=2}^{p-2}\left|\sum_{\chi\neq\chi_0}\chi(a)\,|L(1,\chi)|\right|\right)$$
$$=3\cdot C\cdot p^3+O(p^{\frac{5}{2}}\cdot\ln^2 p).$$

This proves Theorem 2.

Now we prove Theorem 3. For any odd prime p with $p \equiv 3 \bmod 4$ and integer n with $(p, n) = 1$, from Lemma 4 we have

$$\sum_{\chi \neq \chi_0} |G(n, \chi; p)|^6 \cdot |L(1, \chi)| = \sum_{\substack{\chi \neq \chi_0 \\ \chi(-1)=1}} \left[2p + \left(\frac{n}{p}\right) G(1; p) \sum_{a=1}^{p-1} \chi(a) \left(\frac{a^2-1}{p}\right) \right]^3 |L(1, \chi)| \tag{9}$$

and

$$\begin{aligned} \sum_{\chi \neq \chi_0} |G(n, \chi; p)|^6 \cdot |L(1, \chi)| &= \sum_{\chi \neq \chi_0} |\overline{G(n, \chi; p)}|^6 \cdot |\overline{L(1, \chi)}| \\ &= \sum_{\chi \neq \chi_0} \left| \sum_{a=1}^{p-1} \bar{\chi}(a)\, e\left(\frac{-na^2}{p}\right) \right|^6 \cdot |L(1, \bar{\chi})| \\ &= \sum_{\chi \neq \chi_0} \left| \sum_{a=1}^{p-1} \chi(a)\, e\left(\frac{-na^2}{p}\right) \right|^6 \cdot |L(1, \chi)| \\ &= \sum_{\substack{\chi \neq \chi_0 \\ \chi(-1)=1}} \left[2p + \left(\frac{-n}{p}\right) G(1; p) \sum_{a=1}^{p-1} \chi(a) \left(\frac{a^2-1}{p}\right) \right]^3 \cdot |L(1, \chi)|. \end{aligned} \tag{10}$$

Note that $(\frac{-n}{p}) = -(\frac{n}{p})$, from (7), (8), (9), (10), Lemma 1, Lemma 2, and Lemma 3 we obtain the asymptotic formula

$$\begin{aligned} &\sum_{\chi \neq \chi_0} |G(n, \chi; p)|^6 \cdot |L(1, \chi)| \\ &= \frac{1}{2} \left[\sum_{\chi \neq \chi_0} |G(n, \chi; p)|^6 \cdot |L(1, \chi)| + \sum_{\chi \neq \chi_0} |\overline{G(n, \chi; p)}|^6 \cdot |\overline{L(1, \chi)}| \right] \\ &= \frac{1}{2} \sum_{\substack{\chi \neq \chi_0 \\ \chi(-1)=1}} \left[2p + \left(\frac{n}{p}\right) G(1; p) \sum_{a=1}^{p-1} \chi(a) \left(\frac{a^2-1}{p}\right) \right]^3 |L(1, \chi)| \\ &\quad + \frac{1}{2} \sum_{\substack{\chi \neq \chi_0 \\ \chi(-1)=1}} \left[2p - \left(\frac{n}{p}\right) G(1; p) \sum_{a=1}^{p-1} \chi(a) \left(\frac{a^2-1}{p}\right) \right]^3 |L(1, \chi)| \\ &= \sum_{\substack{\chi \neq \chi_0 \\ \chi(-1)=1}} \left[(2p)^3 + 6pG^2(1; p) \left(\sum_{a=1}^{p-1} \chi(a) \left(\frac{a^2-1}{p}\right) \right)^2 \right] \cdot |L(1, \chi)| \\ &= 10 \cdot C \cdot p^4 + O(p^{\frac{7}{2}} \cdot \ln^2 p). \end{aligned}$$

This completes the proof of Theorem 3.

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